converges only when $x \gtreqless 1$. Moreover, the given series is not convergent, since

$$\operatorname*{Lt}_{n \doteq \infty} \frac{(e-1)^n}{n} \neq 0,$$

$e - 1$ being greater than unity.

Also solved by HORACE OLSON, C. E. FLANAGAN, J. W. CLAWSON, and H. S. UHLER.

GEOMETRY.

449. Proposed by H. E. TREFETHEN, Colby College.

Find a tetrahedron with the face angles at one vertex in arithmetical progression and its six edges expressed in positive integers.

SOLUTION BY THE PROPOSER.

Let the angles be $A + B$, A, $A - B$; the lateral edges x, y, z; the base edges a, b, c; so that $a^2 = x^2 + y^2 - 2xy \cos(A + B)$, $b^2 = x^2 + z^2 - 2xz \cos A$, $c^2 = y^2 + z^2 - 2yz \cos(A - B)$. If, for brevity, we put $1 + \cos(A + B) = P/2$, $1 + \cos A = Q/2$, and $1 + \cos(A - B) = R/2$, we may write

$$a^2 = (x + y)^2 - Pxy = (x + y - p)^2 \text{ or } Pxy - 2py = 2px - p^2, \tag{1}$$

$$b^2 = (x + z)^2 - Qxz = (x + z - q)^2 \quad Qxz - 2qz = 2qx - q^2, \tag{2}$$

$$c^2 = (y + z)^2 - Ryz = (y + z - r)^2 \quad Ryz - 2ry = 2rz - r^2. \tag{3}$$

Eliminating y and z from (1), (2), (3), and arranging we have

$$(2px - p^2)[q(4r - Rq) + 2(Rq - Qr)x] = r(Px - 2p)[2q(r - q) + (4q - Qr)x]. \tag{4}$$

If we put the coefficient of $x^2 = 0$, then

$$p = \frac{Pr(4q - Qr)}{4(Rq - Qr)} \tag{5}$$

and also

$$2x = \frac{p^2q(4r - Rq) + 4pqr(q - r)}{p(8qr - Qr^2 - Rq^2) - p^2(Rq - Qr) + Pqr(q - r)}. \tag{6}$$

We may now assign values to A and B and thus determine P, Q, R. If values are assigned to q and r also, p is defined by (5), and then x may be found from (6), y and z from (1), (2), (3) and finally a, b, c also.

Thus if $A = 90°$ and $B = \text{arc-sin } 1/3$, then $P = 4/3$, $Q = 2$, $R = 8/3$; and if also $q = r = 1$, then $p = 1$, $x = 1/4$, $y = 3/10$, $z = 1/3$. Reducing the values of x, y, z to a common denominator and rejecting it, we have in integers $x = 15$, $y = 18$, $z = 20$, and consequently $a = 27$, $b = 25$, $c = 22$. Or since the equations are symmetrical we may use the reciprocals $x = 4$, $y = 10/3$, $z = 3$, whence in integers $x = 12$, $y = 10$, $z = 9$, and then $a = 18$, $b = 15$, $c = 11$. Again if $\sin B = 1/2$, the angles are 120°, 90°, and 60°; $x = 9$, $y = 15$, $z = 40$; $a = 21$, $b = 41$, $c = 35$. If $\sin B = 2/3$, we find $x = 1{,}092$, $y = 416$, $z = 81$; $a = 2{,}804$, $b = 2{,}185$, $c = 367$.

By varying q, r and the $\sin B$, other sets of numbers may be found ad libitum.

464. Proposed by FRANK R. MORRIS, Glendale, Calif.

The sum of the hypotenuse and one side of a right triangle is 100 feet. A point on the hypotenuse is 10 feet from each of the sides. Find the length of the hypotenuse correct to the third decimal place.

SOLUTION BY W. W. BURTON, Mercer University, Macon, Ga.

Let $BC = x$. Then $AB = 100 - x$ and $BD = x - 10$. The right triangles ACB and PDB are similar. (Their sides are respectively parallel.) Therefore

$$AB : BC = PB : BD \text{ or } (100 - x) : x = PB : (x - 10),$$

and

$$PB = \frac{(x-10)(100-x)}{x}.$$

Again

$$\overline{PB}^2 = \overline{PD}^2 + \overline{BD}^2 \text{ or } \overline{PB}^2 = 10^2 + (x-10)^2 \text{ or } x^2 - 20x + 200.$$

Therefore

$$PB = \sqrt{x^2 - 20x + 200},$$

and

$$\frac{(x-10)(100-x)}{x} = \sqrt{x^2 - 20x + 200}.$$

Reducing, we get the cubic,

$$2x^3 - 139x^2 + 2200x - 10000 = 0.$$

Applying Descartes' rule of signs we find that all of the roots of the above cubic are positive. Applying Sturm's theorem we find that the roots are situated as follows: one root between 9 and 10; one root between 11 and 12; and one root between 49 and 50.

The root between 9 and 10 must be discarded, as it is impossible in our problem. Applying Horner's method for incommensurable roots, we find the roots to be 11.282 and 49.212 correct to three decimal places. Therefore $BC = 11.282$ or 49.212. Hence, the hypotenuse can be *88.718* ft. or *50.788* ft.

Also solved by HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, M. HELEN KELLEY, C. E. FLANAGAN, W. E. WHITFORD, G. H. HARTWELL, and NATHAN ALTSHILLER.

465. Proposed by ROGER A. JOHNSON, Western Reserve University.

Let C be a fixed circle, A a point outside it. Let AT and AT' be the tangents from A to the circle, touching the latter at T and T'. Let two secants be drawn through A, cutting the circle at P, Q and R, S respectively. Let PR and QS meet at X, PS and QR meet at Y. Prove by elementary methods that for all positions of the secants, X and Y lie on the line TT'.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

I. $PQSR$ is a quadrangle, having X, Y, A for its diagonal points. Hence $X(ARYS)$ is a harmonic pencil. Then, if XY cuts ARS at Z, $(ARZS)$ is a harmonic range.

Now TT' is the polar of A with respect to the circle whose center is C. Hence, if TT' cuts ARS at Z', $(ARZ'S)$ is a harmonic range.

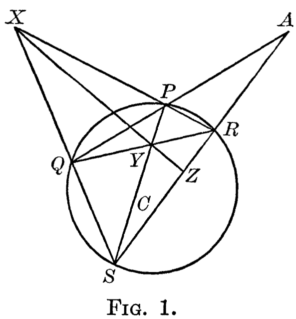

FIG. 1.

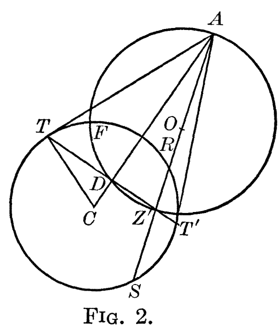

FIG. 2.

Therefore, the points Z and Z' coincide and XY and TT' cut the line ARS at the same point. Similarly, it can be shown that XY and TT' cut the line APQ at the same point. Hence the lines XY and TT' coincide.

[The exercise follows at once from the theorem: If a system of conics circumscribe a given quadrangle, the diagonal point triangle is a self-conjugate triangle w. r. t. each conic of the system. (Durell, Plane Geometry for Advanced Students, Part ii, p. 110.)]